Lagrange Points

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This document explains Lagrangian points (a.k.a. Lagrange points, L-points or libration points) in a simple manner. Lagrangian points are the specific points where -when put- a relatively light object can maintain its position relative to the other two heavy celestial bodies (one of which rotate around the other one) in space. Sun and Earth can be used to conceptualize the large celestial bodies, and satellites as light objects (relatively). The main contribution on the light object's motion comes from the gravitational forces which arise from the two heavy bodies. However, when switched to rotating frame of reference for convenience, some other factors (e.g. Coriolis force) have effect on the object at different Lagrangian points. In this paper, the most significant properties of Lagrange points are to be discussed, such as the stability conditions of different Lagrangian points, mathematical derivations of them, their discovery, relation to three body problem and why these points are being used in spaceflight applications. Unless otherwise is specified, celestial bodies are to be considered by default throughout the article.

DEFINITION OF LAGRANGE POINTS

Consider two large celestial bodies (named as B_1, B_2) orbiting around each other and a relatively small object S (Fig. 1). According to Newton's law of universal gravitation, every two particle (in any size) attracts each other with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them [1]. A Lagrange point is a position of the small object S in space, where it maintains its position relative to B_1 and B_2, mainly due to the gravitational forces exerted onto it by B_1 and B_2. For any two large celestial bodies orbiting, there exists exactly 5 Lagrange points. At any of these points, the small object orbits in an unchanging pattern with the two heavy bodies. As a requisite of this condition, the small object orbits with the same period which the large objects rotate around each other. The restriction of the body which is to be put at one of the Lagrange Points near two large bodies being small (more accurately, being light) is due to the force it exerts onto the large bodies. This force it exerts onto large bodies is negligible for them because of their large masses. By Newton's 3$^{\text{rd}}$ Law of Motion, the force exerted by any of the two objects onto another is equal in magnitude [1], e.g., $F_{B_1 \to C} = F_{C \to B_1}$. Thus, the same amount of force is exerted onto it; however, this force has a great effect on it because of its small mass. This is a direct result of Newton's 2$^{\text{nd}}$ Law of Motion [1]. A more detailed and a mathematical explanation – derivation will be provided in the upcoming sections.

HISTORICAL BACKGROUND - THREE-BODY PROBLEM

The conceptual starting point of Lagrange Points is the Three-body Problem (more specifically, Euler's Three-body Problem), put forward by the Swiss mathematician, physicist, astronomer, logician and engineer Leonhard Euler, who lived in the 18$^{\text{th}}$ Century [2]. Even we state it directly as Three-body Problem, there exists types of Three-body Problem which are nothing but the restricted cases of the General Three-body Problem. Before investigating these types in detail, one should note that the General Three-body Problem is still unsolved despite the development of mathematical calculation methods and enhanced computers, because no closed-form solution exists for all sets of initial conditions. Hence, the best solution one can obtain is a numerical one, and even in that case solution is likely to be chaotic. [3].

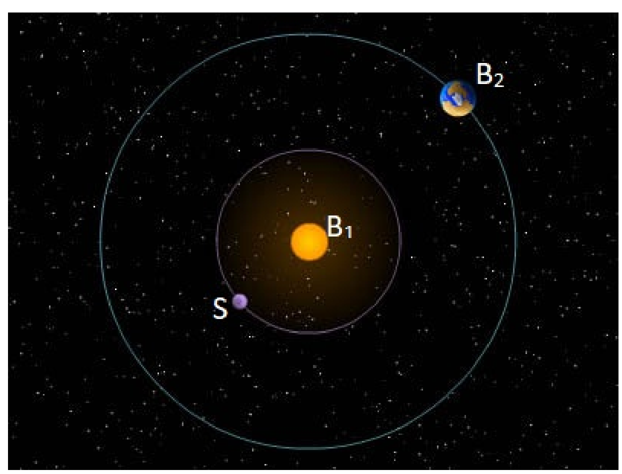

FIG. 1: B_1: Sun, B_2: Earth, S: Satellite
Drawing is not to scale.

General Three-body Problem

To start with, let us state the General Three-body Problem. Consider the three masses m_1, m_2, m_3 and the vectors $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$ denoting their positions, respectively. If we are to write Newtonian equation of motion for this system we get:

$$\ddot{\mathbf{x}}_{\mathrm{i}} = -Gm_j\frac{\mathbf{x_i}-\mathbf{x_j}}{|\mathbf{x_i}-\mathbf{x_j}|^3} - Gm_k\frac{\mathbf{x_i}-\mathbf{x_k}}{|\mathbf{x_i}-\mathbf{x_k}|^3} \quad (1)$$

Now, one can switch to center of mass (CM) coordinate system[1]. for convenience. At the center of mass coordinate system, as the name implies, center of the total body's mass is at the center of the coordinate system we have switched to. Thus, for all the cases we use the CM coordinate system we get the following equations:

$$\sum_{i=1}^{3} m_{\mathrm{i}}\mathbf{x}_{\mathrm{i}} = 0 \quad , \quad \sum_{i=1}^{3} m_{\mathrm{i}}\dot{\mathbf{x}}_{\mathrm{i}} = 0 \quad (2)$$

where the equation on the RHS is nothing but the derivative of the one at the LHS.

At this point, it is necessary to note that we assume that there is not any external force nor torque effecting on the system, which implies energy and angular momentum are conserved. Also, some more eliminations can be made, such as eliminating time and fixing the line of nodes, which we will not get into detail here. In the end, it is possible to reduce the order of the system to 4 at most, which still makes the problem unsolvable. [5]

It is possible to write the equations of motion in a more symmetrical manner by denoting the positions of the masses with the vectors that show the relative positions of the particles to each other, as it is depicted in the Fig. 2. Accordingly, Eq. (1) and Eq. (2) turns into:

$$\ddot{\mathbf{s}}_{\mathrm{i}} = -GM\frac{\mathbf{s_i}}{{s_{\mathrm{i}}}^3} + m_i\mathbf{G} \quad (3)$$

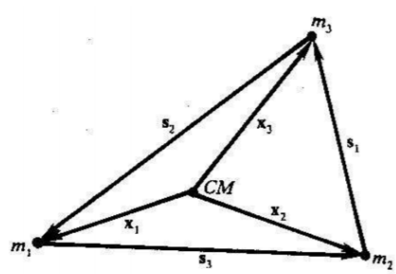

FIG. 2: Position vectors in the CM system and relative position vectors for the three-body problem. [4]

[1] One can refer to S. Thornton and J. Marion, *Classical Dynamics of Particles and Systems* (Brooks/Cole, 2004) for details of changing coordinate system and CM coordinate system.

Euler's Solution to Three-body Problem

Second, let us consider the Euler's solution to the General Three-body Problem. Euler's solution considers the solution of the Three-body Problem for the masses which are colinear, i.e., masses that lie on the same line. For any two points, there is always a line passing through them. However, since there are three points (masses) in the Three-body Problem, putting them on a line restricts the solutions. Therefore, the Euler's solution gives only three points, which is to be generalized to 5 by Lagrange in the following years after Euler published his solution.

Let m_2 to be mass between the other two masses, without loss of generality. Therefore, $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$ become multiples of each other since they are vectors with same / different lengths at the same or the opposite directions. This relation can be expressed in the form:

$$\mathbf{s}_1 = \lambda\mathbf{s}_3 \quad , \quad \mathbf{s}_2 = -(\lambda+1)\mathbf{s}_3 \quad (4)$$

Therefore, we can express all vectors in terms of scalar multiples of one of them. Then, we can write the Newtonian equation of motion. As a result, Eq. 3 turns into:

$$\ddot{\mathbf{s}}_3 = -\frac{m_2 + m_3(1+\lambda)^{-2}}{m_2 + m_3(1+\lambda)}\frac{GM\mathbf{s}_3}{{s_3}^3} \quad (5)$$

Therefore, we get the solution that the masses move along confocal ellipses, which have the same eccentricity and same orbital period around CM. Importantly, they are always lined up. It has not been discussed yet, but in the section on stability, it will be referred briefly that Euler's solutions are unstable. Therefore, any small perturbation corrupts the system. However, it is still much more easier for small objects to maintain their position on Euler's solutions rather than trying to do it on any arbitrary point.

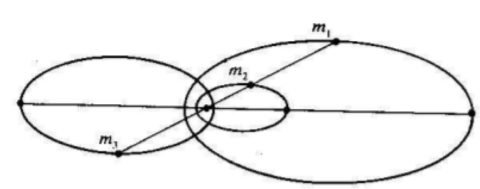

FIG. 3: Collinear solutions. Masses m_1, m_2, m_3 are all on the same line. [4]

Lagrange's Solution to Three-body Problem

Finally, we consider Lagrange's solution. Joseph-Louis Lagrange was an Italian astronomer and mathematician, who was also the successor of Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, Prussia [6]. He considered the case where one of the masses in Three-body Problem was much lighter than the other two. As a result, in addition to Euler's three colinear solutions, he found out two other equilateral solutions for any three masses with circular orbits.

The equilateral case is equivalent to setting $\mathbf{G}$ in Eq. (3) to 0. If $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$ form a equilateral triangle, the vectoral summation gives this result since all $|\mathbf{s}_\mathrm{i}|$ have the same magnitude and the overall shape is a closed loop. As a result, he decoupled the equations for $\mathbf{s}_\mathrm{i}$. For bounded cases, these decoupled equations have solutions that trace ellipses. The conclusion of his attempts is that when the conditions he set are satisfied, particles follow an ellipse of the same eccentricity with their common center of mass located at the focal point of all three orbits. Thus, all masses have the same period while they keep up the shape of an equilateral triangle even the triangle changes size and rotates.

In contrast to Euler's solutions, the points Lagrange found out are at the stable equilibrium, with the restriction that one of the masses must be much heavier than the other two. Fig. 1 is an example of this case, where the mass of the sun is much greater than the other two masses. However, one should notice that the masses in Fig. 1 are colinear, which is an example of one of the three points found out by Euler, not Lagrange. The stability conditions of these points are to be examined later in this paper.

Even Euler found out the first three solutions to Three-body Problem, in general, these five special solutions are named after Lagrange. So, when Lagrange Points are considered, one refers to the five of the solutions actually discovered by both Euler and Lagrange. To be more specific, these points are named as L_1, L_2, L_3, L_4 and L_5. These Lagrange points are indicated on the Fig. 4 for the Sun-Earth system, though one should remind herself that Sun-Earth system is not the only case that fits Lagrange's Three-body Problem solution.

EQUATIONS AND EXPLANATIONS OF EACH OF THE LAGRANGE POINTS

Consider two massive objects in orbits around their center of mass. Since we are working with Lagrange Points, we always assume that one of the masses is always much larger than the other large mass (how large it should be is not a fixed number for all points and should be considered separately). Now, we add a third body, who has a negligible mass when compared to the other

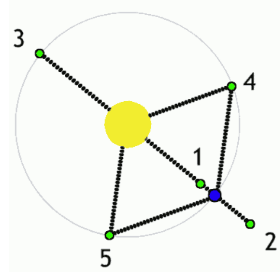

FIG. 4: Lagrange Points in L_1, L_2, L_3, L_4, L_5 in the Sun - Earth system.

two massive bodies. We are now to examine the points where when we put this body, it stays relatively stationary with respect to the massive bodies, i.e., it will have the same periodicity (thus, angular velocity) as them.

When working with Lagrange Points, it is convenient to switch to a rotating coordinate system. We have stated that two massive objects orbit around their center of mass. If we set our coordinate system rotating with the same angular velocity Ω as they do and set its origin to the center of mass of the two large bodies, then one sees them motionless. This is what we want; moreover, we want to find a position where the small mass can stay motionless too. Since we have switched to a rotating frame, we will have pseudo-forces, so effects of centrifugal and coriolis forces must be taken into consideration. Especially, they will become important when we examine the stability conditions of the Lagrange Points. Then, instead of the the equation of force 1, we use the equation of effective force in a frame rotating with angular velocity $\overrightarrow{\Omega}$:

$$\overrightarrow{F}_\Omega = \overrightarrow{F} - 2m(\overrightarrow{\Omega} \times \dot{\overrightarrow{r}}) - m\overrightarrow{\Omega} \times (\overrightarrow{\Omega} \times \overrightarrow{r}) \qquad (6)$$

where the first correction is the Coriolis force and the second one is the centrifugal force. This equation can be derived from the generalized potential:

$$U_\Omega = U - \overrightarrow{\nu} \cdot (\overrightarrow{\Omega} \times \overrightarrow{r}) + \frac{1}{2}(\overrightarrow{\Omega} \times \overrightarrow{r}) \cdot (\overrightarrow{\Omega} \times \overrightarrow{r}) \qquad (7)$$

where a contour plot of it can be seen in the Fig. 5, which might be helpful for visualization.

In the case where z axis of the cartesian coordinate system is aligned with the angular velocity, we have:

$$\overrightarrow{\Omega} = \Omega \hat{k} \qquad (8)$$

$$\overrightarrow{r} = x(t)\hat{i} + y(t)\hat{j} \qquad (9)$$

$$\overrightarrow{r_1} = -\alpha R\hat{i} \qquad (10)$$

$$\overrightarrow{r_2} = \beta R\hat{i} \qquad (11)$$

where $\alpha = \left(\frac{M_2}{M_1+M_2}\right)$, $\beta = \left(\frac{M_1}{M_1+M_2}\right)$ in the above equations.

We want to find static equilibrium positions, so we set velocity $\mathbf{v} = \dot{\vec{r}}$ to zero and seek for solutions. Using symmetry about the x-axis, for the case $\alpha << 1$, with approximation we get the first three Lagrange Points:

$$L_1 : \left(R[1 - (\frac{\alpha}{3})^{\frac{1}{3}}], 0\right) \tag{12}$$

$$L_2 : \left(R[1 + (\frac{\alpha}{3})^{\frac{1}{3}}], 0\right) \tag{13}$$

$$L_3 : \left(-R[1 + \frac{5}{12}\alpha], 0\right) \tag{14}$$

To find L_4 and L_5, one needs to balance the centrifugal forces acting upon them. Centrifugal force is in the direction of the vector starting at the origin and pointing outward. Gravitational forces are to be used to balance the centrifugal force. As a result, when the force acted on the object at L_4 and L_5 are seperated into its components, we expect to observe that the net force acting perpendicular (to the direction of rotation) and the net force acting parallel (to the direction of rotation) are zero. These correspond to the following equations:

$$F_{\Omega}{}^{\perp} = \alpha\beta y \Omega^2 R^3 \left(\frac{1}{((x - R\beta)^2 + y^2)^{\frac{3}{2}}} - \frac{1}{((x + R\alpha)^2 + y^2)^{\frac{3}{2}}}\right) \tag{15}$$

$$F_{\Omega}{}^{\parallel} = \Omega^2 \frac{x^2 + y^2}{R} \left(\frac{1}{R^3} - \frac{1}{((x - R\beta)^2 + y^2)^{\frac{3}{2}}}\right) \tag{16}$$

Setting 15 and 16 equal to 0, we get the remaning two Lagrange points:

$$L_4 : \left(\frac{R}{2}\left(\frac{M_1 - M_2}{M_1 + M_2}\right), \frac{\sqrt{3}}{2}R\right) \tag{17}$$

$$L_5 : \left(\frac{R}{2}\left(\frac{M_1 - M_2}{M_1 + M_2}\right), -\frac{\sqrt{3}}{2}R\right) \tag{18}$$

STABILITY STATES OF THE LAGRANGE POINTS

From the effective potential contour plot Fig. 5, it is usually clear which Lagrange points are stable and which are not. We consider the hills, valleys and saddles at the plot. However, it may mislead one. So, following the standard procedure, which is perturbing each equilibrium solution with a small amount, gives us the reliable information. Thus, we replace x, y, z by their perturbed versions with small amounts:

$$x = x_{\mathrm{i}} + \delta_{\mathrm{x}} \tag{19}$$

$$y = y_{\mathrm{i}} + \delta_{\mathrm{y}} \tag{20}$$

$$z = \delta_{\mathrm{z}} \tag{21}$$

where $x_{\mathrm{i}}, y_{\mathrm{i}}$ represent the $i^{\text{ th}}$ Lagrange Points and $\delta_{\mathrm{x}}, \delta_{\mathrm{y}}, \delta_{\mathrm{z}}$ are small perturbations about these points.

Oscillatory or decaying solutions can be interpreted as stable solutions; whereas, exponentially diverging solutions lead to unstability. When the x, y, z stated in the Eq. 19 are put into the Eq. 7 (Calculations won't be included in here for simplicity. The reader can refer to the document The Lagrange Points [7] by Cornish for detailed calculations.), L_1, L_2 and L_3 turns out to be saddle points, where we get positive real roots meaning that small perturbations lead to exponential growth (unstable). The outcome is different for L_4 and L_5. As a result, we get pure imaginary solutions, which implies stability as long as the ratio of two heavy masses exceeds 24.96 [7]. These points turn out to be local maxima points, which usually imply unstable equilibrium. However, due to coriolis force, these points are actually stable. When a mass situated at L_4 and L_5 perturbed with small amount, it tends to slide down the potential. When it does so, its speed increases and the Coriolis force pushes it back to the equilibrium point.

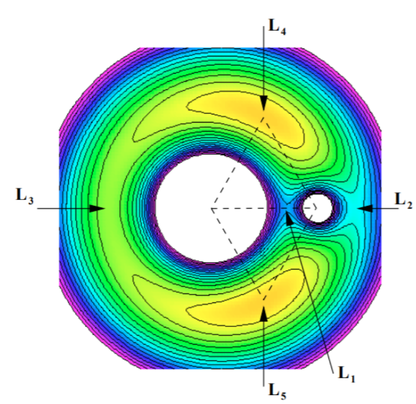

FIG. 5: Generalised potential - contour plot [7]

Therefore, it is expected in the universe to observe objects at the L_4 and L_5 points for different celestial object couples. In fact, this is a common situation that these object are named as *Trojans*. Some of the natural objects found at L_4 and L_5 points for different celestial object couples are:

- Sun-Jupiter system has trojans, named after the famous Greek poet Homer's *Iliad*.
- Sun-Neptun system has many trojans at L_4 and L_5.
- Sun-Earth system contains interplanetary dust and asteroids located at L_4 and L_5.
- Earth-Moon system contain interplanetary dust, named as *Kordylewski clouds* at L_4 and L_5.
- Saturn-Tethys (Saturn's moon) system contains two smaller moons located at L_4 and L_5 points.

Although we have stated that L_1, L_2 and L_3 are unstable equilibrium points and they may seem useless when there are L_4 and L_5 which are completely stable, it is not the case in real life applications. When space missions carried out by NASA, ESA and CNSA are considered, in fact, one observes that L_1 and L_2 are in high demand for Sun-Earth and Earth-Moon systems. Also, future missions are mostly planned to send satellites at L_1 and L_2 points. But why is it preferred to send satellites to unstable equilibrium points? In fact, it is true that L_1,L_2,L_3 are unstable equilibrium points, but they are nonetheless equilibrium points. So, they are as practical as L_4 and L_5 as long as there is no perturbation. Even there is small perturbations, the cost of keeping them around the equilibrium point is almost insignificant. Therefore, for some other advantages of L_1, L_2 (such as quasi-periodic Lissajous orbits[2] at these points, which is why space missions mostly consider them as practical), the cost is insignificant. As an example, the popular James Webb Space Telescope is planned to be sent to L_2 in the Sun-Earth system.

CONCLUSIONS

This paper provides the basic ideas and derivations relating to Lagrange Points by considering the historical development of Three-body Problem, Lagrange's contributions and how these special points are being used for space applications. Rather than going through all the calculations and examples, the notion behind each step that leads to the solution and the important examples to comprehend the subject have been included. In depth calculations and topics related to the subject but not necessary to comprehend it are provided in the references for further reading. The most important aim had been to clearly explain what Lagrange Points are, how they were discovered and for what applications they are being used today. I thank my PHYS201: Classical Mechanics teacher Fethi M. Ramazanoğlu, who taught me the wonders of Physics and has always tried to arouse his students' curiosity.

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[2] Quasi-periodic Lissajous orbits are not to be covered in this paper since it is far beyond the subject of this introductory paper and not necessarily important to understand Lagrange points. However, the reader can review the exquisite paper *Quasi-Periodic Orbits of the Restricted Three-Body Problem Made Easy* [8] by Kolemen, Kasdin and Gurfil on the subject.